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Clustering in one-dimensional threshold voter models

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We consider one-dimensional spin systems in which the transition rate is 1 at site k if there are at least N sites in $\{k-N, k-N+1, \ldots, k+N-1, k+N\}$ at which the 'opinion' differs from that at k, and the rate is zero otherwise. We prove that clustering occurs for all $N \geq 1$ in the sense that $P[\eta_t(k) \neq \eta_t(j)]$ tends to zero as t tends to ∞ for every initial configuration. Furthermore, the limiting distribution as $t \to \infty$ exists (and is a mixture of the pointmasses on $\eta \equiv 1$ and $\eta \equiv 0$) if the initial distribution is translation invariant. In case $N = 1$, the first of these results was proved and a special case of the second was conjectured in a recent paper by Cox and Durrett.

Now let $D(\rho)$ be the limiting density of 1's when the initial distribution is the product measure with density ρ. If $N = 1$, we show that $D(\rho)$ is concave on $[0, \frac{1}{2}]$, convex on $[\frac{1}{2}, 1]$, and has derivative 2 at 0. If $N \geq 2$, this derivative is zero.

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1. Introduction

The (linear) voter model has been the subject of a large number of papers during the past fifteen years (see, for example, Chapter V of Liggett, 1985; and Bramson, Cox and Griffeath, 1988; and the references there). It exhibits two basic kinds of behavior: (a) In dimensions one and two it clusters, in the sense that $P[\eta_t(k) \neq \eta_t(j)]$ tends to zero as t tends to ∞ for every initial configuration. (b) In dimensions three and larger, it exhibits coexistence, in the sense that there are invariant measures which concentrate on configurations with infinitely many 0's and 1's. In a recent paper, Cox and Durrett (1991) initiated the study of a class of spin systems which

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they call nonlinear voter models. They discovered that there is coexistence in some of these, even in one dimension. This raises the question of determining for exactly which nonlinear voter models one has coexistence and for which one has clustering. Obtaining a complete answer to this question is probably out of the question. Nevertheless, in this paper we will prove some results which contribute to its answer.

We will consider only a subclass of nonlinear voter models: the threshold-T voter model in one dimension with range N. This is the continuous time Markov process η_t on $\{0,1\}^{\mathbb{Z}}$ with transitions given as follows: the coordinate $\eta(k)$ is replaced by $1-\eta(k)$ at rate

$$c(k,\eta)=\begin{cases}1 & \text{if the number of } j \text{ such that } |j-k|\leq N \text{ and } \eta(j)\neq\eta(k) \text{ is at least } T,\\ 0 & \text{otherwise.}\end{cases}$$

In their paper, Cox and Durrett proved the following results for this class:

(a) Coexistence occurs if $T=1$ and $N\geq 4$. The proof is based on a comparison of the voter model with a contact process. The comparison is such that if the contact process survives, then the corresponding voter model exhibits coexistence. Use of the Holley–Liggett upper bound (Theorem 1.33 of Chapter VI of Liggett, 1985) on the critical value of the basic contact process yields the result for $N\geq 7$. Analogous results obtained in Liggett (1991) for variants of the contact process give the improvement down to $N\geq 4$.

(b) Coexistence occurs if $T=2$ and $N\geq 47$ (Theorem 8).

(c) Clustering occurs if $T=N=1$ (Theorem 4).

Cox and Durrett also conjectured that coexistence occurs if $T=1$ and $N=2$ or 3 (Conjecture 1). Durrett (1992) proved that for each $a\in(0,\frac{1}{2})$, there is coexistence for the process with $T=[aN]$ if N is sufficiently large. It is easy to see that clustering does not occur if $T>N$, since then the configurations in which strings of length $\geq N+1$ of 0's and 1's alternate are traps for the process. (Durrett and Steif have noted that in this case, the process 'gets stuck' for any initial distribution.) In view of these remarks, the case $T=N$ appears to be of particular interest. We now restrict ourselves to this case.

We need some notation in order to state our results. If μ is a probability measure on $\{0,1\}^{\mathbb{Z}}$, then μ_t will denote the distribution at time t when the initial distribution is μ. The product measure with density $\rho\in[0,1]$ will be denoted by v_ρ. We then have the following:

Theorem 1. *Suppose $T=N\geq 1$.*

(a) *If η is any initial configuration and k and j are any two sites, then*

$$\lim_{t\to\infty} P^\eta[\eta_t(k)\neq\eta_t(j)]=0.$$

(b) *If μ is translation invariant, then*

$$\lim_{t\to\infty}\mu_t=D(\mu)v_1+[1-D(\mu)]v_0,$$

where $D(\mu)$ is some constant depending on the initial distribution μ.

Part (a) of Theorem 1 is Theorem 4 of Cox and Durrett (1991) if $N = 1$. It was conjectured by Durrett for general N in a private communication. Part (b) of Theorem 1 in case $N = 1$ and $\mu = \upsilon_\rho$ is Conjecture 3 of Cox and Durrett (1991).

It would be interesting to 'compute' the limiting density in part (b) of Theorem 1, at least when the initial distribution is the product measure with density ρ. This is probably not possible, but we can compute the derivative of this density at $\rho = 0$. This is the main content of the next two theorems.

Theorem 2. *Suppose $T = N = 1$. Then*

$$\text{(a)} \qquad \lim_{\rho \downarrow 0} \frac{D(\upsilon_\rho)}{\rho} = 2,$$

and

(b) *$D(\upsilon_\rho)$ is concave as a function of ρ on $[0, \frac{1}{2}]$ and is convex on $[\frac{1}{2}, 1]$.*

In the case treated in Theorem 2, the threshold voter model has a dual called a double branching annihilating random walk (DBARW) which has been studied by Sudbury (1990). In this process, particles are distributed on Z with at most one particle per site. A particle at k jumps to $k+1$ and $k-1$ with rate $\frac{1}{2}$ each. In addition, it creates two new particles at rate $\frac{1}{2}$ which are placed at $k+1$ and $k-1$ respectively. Whenever two particles end up at the same site, they annihilate each other, and the site becomes vacant.

We consider the DBARW evolving on the finite subsets of $\mathbb{Z}$, and identify configurations $\eta \in \{0, 1\}^{\mathbb{Z}}$ with subsets $\{k \in \mathbb{Z}\colon \eta(k) = 1\}$. Letting $|A|$ denote the cardinality of A, we have the following statement of the duality relation between the two processes (see Cox and Durrett, 1991):

$$P^{\eta}(\eta_t(k) = 1) = P^{\{k\}}(|A_t \cap \eta| \text{ is odd}). \tag{1.1}$$

Here η_t is the threshold voter model (with $T = N = 1$), and A_t is the DBARW with initial configurations η and $\{k\}$ respectively. Note that if the DBARW starts in a set of odd cardinality, its cardinality remains odd at all later times. In particular, it never becomes empty. We now identify subsets of Z of odd cardinality which differ by translations, and consider the DBARW as evolving on the resulting quotient space.

For a probability measure μ on that state space, define

$$\mu(n) = \sum_{A:\,|A| = n} \mu(A).$$

We can now state a corollary to Theorem 2. The positive recurrence part of the statement was conjectured by Cox and Durrett (1991).

Corollary. *The DBARW on the above quotient space is positive recurrent, and its unique invariant measure π satisfies*

$$\sum_{n=0}^{\infty} (2n+1)\pi(2n+1) = 2.$$

The behavior of the limiting density for larger range is described in the next result. We use a subscript to denote the value of N.

Theorem 3. *Suppose* $T = N$.

(a) $$\lim_{\rho\downarrow 0} \frac{D_N(v_\rho)}{\rho^{N-1}} = 0.$$

(b) $$\liminf_{\rho\downarrow 0} \frac{D_N(v_\rho)}{\rho^{N}} > 0.$$

(c) $$\lim_{N\to\infty} D_N(v_\rho) = 0 \quad \text{if } \rho < \tfrac{1}{2}.$$

Remarks. (i) A simple consequence of part (a) above is that if $N \geq 2$, the threshold voter model does not have a dual satisfying (1.1). If it did, a slight modification of the proof of part (b) of Theorem 2 would imply that the concavity statement there would apply here as well. But that would contradict part (a) of Theorem 3.

(ii) In view of Theorem 2, part (a) of Theorem 3 is only of interest if $N \geq 2$.

(iii) Theorem 3 suggests that the following statement may be true: For $N \geq 2$ and $\rho < \frac{1}{2}$, $D_N(v_\rho)$ is decreasing in N and convex in ρ. It would be interesting to prove either part.

(iv) With some additional work, one can improve part (a) of Theorem 3 to show that $D_N(v_\rho)$ is $O(\rho^N)$ as $\rho \downarrow 0$. Part (b) of Theorem 3, which shows that this improvement is best possible, is due to an observation of Durrett.

Theorem 1 is proved in the next section. Theorem 2 and its corollary are proved in Section 3. Theorem 3 is proved in Section 4.

2. Threshold = Range

This section is devoted to the proof of Theorem 1. We begin with the following definitions:

Definition. We say that x is *flippable* at $\eta \in \{0, 1\}^{\mathbb{Z}}$ if $c(x, \eta) = 1$.

Definition. Given η in $\{0, 1\}^{\mathbb{Z}}$ and A a subset of $\mathbb{Z}$, η^A will denote the state which agrees with η on A^c but disagrees with it at every site of A. The state $\eta^{0,A}$ will equal η on A^c and be identically 0 on A. The state $\eta^{1,A}$ is similarly defined.

Definition. We say that ξ is *flippable from* η *on an interval* I if there exist finite sequences $\eta = \eta_0, \eta_1, \ldots, \eta_m = \xi$ in $\{0, 1\}^{\mathbb{Z}}$ and $y_0, y_1, \ldots, y_{m-1}$ in I so that

(i) $$\eta_j = \eta_{j-1}^{\{y_{j-1}\}}$$

and

(ii) $$c(y_{j-1}, \eta_{j-1}) = 1.$$

Lemma. *Let I be any interval. Any configuration η is flippable on I to at least one of $\eta^{0,I}$ and $\eta^{1,I}$.*

Proof. We prove the result by induction on $v(\eta)$, the number of maximal subintervals of I on which η is constant. The result is clear if $\eta=\eta^{0,I}$ or $\eta^{1,I}$. Suppose now that the result is true for all ζ with $v(\zeta)<v$ and that $v(\eta)=v>1$. There exist subintervals $[n, m]$ and $[m+1, r]$ of I which are maximal, on which η is constant and such that $\eta(m)$ is not equal to $\eta(m+1)$. Assume without loss of generality that $\eta(m)=1$. Now since $\eta(m)=1$ and $\eta(m+1)=0$ it follows that

$$\sum_{0<|j-m|\leq N} \eta(j) \leq \sum_{0<|j-(m+1)|\leq N} \eta(j).$$

This means that either m or $m+1$ is flippable for η. Furthermore, if m is flippable for η, then $m-1$ is flippable for $\eta^{\{m\}}$, $m-2$ is flippable for $\eta^{\{m-1,m\}}, \ldots, n$ is flippable for $\eta^{\{[n-1,m]\}}$. A similar statement holds if $m+1$ is flippable. Hence there is a ζ which is flippable from η and which has $v(\zeta)<v$. So by induction, we find that η is flippable to $\eta^{1,I}$ or $\eta^{0,I}$. $\square$

Proof of part (a) of Theorem 1. We will assume that our threshold voter model is constructed by Harris' method from a family of independent rate one Poisson processes $\{K_t(x), x \in \mathbb{Z}\}$, as described in Cox and Durrett (1991). We will need two additional rate one Poisson processes which are independent of the others, and will be denoted by $K_t(-\infty)$ and $K_t(\infty)$. The set of event times for the Poisson processes will be denoted by $K(x)$. We may also assume without loss of generality that $k=1$ and $j=0$. We will say that ζ in $\{0,1\}^{\mathbb{Z}}$ is in state A (for 'agree') if $\zeta(i)$ are all equal for $|i|\leq N$. By the lemma, for every η there exists an $m(\eta)$ in $\mathbb{Z}_+$ and $y_0(\eta), y_1(\eta), \ldots, y_{m-1}(\eta)$ in $[-N, N]$ which are measurable functions of $\{\eta(i); |i|\leq 2N\}$ and such that (for $m>0$) for $0\leq j<m$,

$$c(\eta^{\{y_0,\ldots,y_{j-1}\}}, y_j)=1,$$

and

$$\eta^{\{y_0,\ldots,y_{m-1}\}}$$

is constant on $[-N, N]$. We take M to be the largest value $m(\eta)$ assumes and for each η we define $y^i(\eta)$ for i in $[0, M-1]$ by

$$y^i(\eta)=\begin{cases} y_i(\eta) & \text{for } i<m(\eta),\\ 0 & \text{for } i\geq m(\eta).\end{cases}$$

Now for the threshold voter model beginning at any η, if the first M arrivals among the Poisson processes $K_t(x)$, $|x|\leq 2N$ occur in order at $y^0(\eta), y^1(\eta), \ldots, y^{M-1}(\eta)$, then by the time of the last of these arrivals, the threshold voter model will be at a configuration in state A. More generally, if at time t the threshold voter model has value η_t and the next M arrivals among the Poisson processes $K_t(x)$, $|x|\leq 2N$ occur in order at $y^0(\eta_t), y^1(\eta_t), \ldots, y^{M-1}(\eta_t)$, then at the time of the last of these flips the threshold voter model will be at a configuration in state A.

For our threshold voter model and a (possibly random) time T, we say the event $A^{1,T}$ occurs if the first M arrivals after T among the Poisson processes $K_t(x)$, $|x| \leq 2N$ occur in order at $y^0(\eta_T), y^1(\eta_T), \ldots, y^{M-1}(\eta_T)$. The stopping time (with respect to the natural filtration of the Poisson processes) $V(1, T)$ is defined to be the time at which $A^{1,T}$ occurs or becomes impossible. We define for $n > 1$,

$$V(n, T) = V(1, V(n-1, T)) \quad \text{and} \quad A^{n,T} = A^{1,V(n-1,T)}.$$

Note that if T is a stopping time (with respect to the Poisson process filtration), then $V(1, T) - T$ is independent of F_T, and in particular, of η_T. Similarly the event $A^{n,T}$ is independent of the sigma-field $F_{V(n-1,T)}$. If

$$Q(T) = \inf\{n \colon A^{n,T} \text{ occurs}\},$$

then the random variable $V(Q(T), T) - T$ is independent of F_T and has finite mean. The random variable $V(Q(T), T)$ is a stopping time, at which point the threshold voter process will be at a configuration in state A. It should be noted that $V(Q(T), T)$ is by no means the first time after T that our process is at a configuration in state A; η_T may well be in state A.

Suppose now that S is a stopping time so that a.s. η_S is in state A. We define the continuous time simple random walks $R_{i,S}(\cdot)$ for $i = -1$ and $i = +1$ as follows:

$$R_{i,S}(0) = iN,$$

$$\text{if } S + t \in K(R_{i,S}(t-) + i) \text{ and } \eta_{S+t}(R_{i,S}(t-)) \neq \eta_{S+t}(R_{i,S}(t-) + i),$$

$$\text{then } R_{i,S}(t) = R_{i,S}(t-) + i,$$

$$\text{if } S + t \in K(i\infty) \text{ and } \eta_{S+t}(R_{i,S}(t-)) = \eta_{S+t}(R_{i,S}(t-) + i),$$

$$\text{then } R_{i,S}(t) = R_{i,S}(t-) + i,$$

$$\text{if } S + t \in K(R_{i,S}(t-)), \quad \text{then } R_{i,S}(t) = R_{i,S}(t-) - i,$$

and these are the only jumps of the random walks. Let $\sigma(S)$ be the stopping time (for the above random walks)

$$\sigma(S) = \inf\{t \colon |R_{i,S}(t)| < N \text{ for } i = -1 \text{ or } i = +1\}.$$

Note that $\sigma(S)$ is independent of S, that $S + \sigma(S)$ is a stopping time for the natural filtration, and that throughout the time interval $[S, S + \sigma(S))$, the process η_t is in state A. Finally note that $P[\sigma(S) > t]$ is of the order $1/t$ for large t and so in particular $E[\sigma(S)]$ is infinite. To complete the proof we note that if we define the stopping times

$$T_1 = V(Q(0), 0)$$

and

$$S_1 = \sigma(T_1) + T_1,$$

and for $n > 1$,

$$T_n = V(Q(S_{n-1}), S_{n-1})$$

and

$$S_n = \sigma(T_n) + T_n,$$

then the random variables $\{T_n - S_{n-1}\}, \{S_n - T_n\}$ are independent, the first set of these random variables are identically distributed with finite mean, and the second set are also identically distributed with infinite mean. This implies that

$$0, T_1, S_1, \ldots, T_n, S_n, \ldots,$$

is an alternating renewal sequence. It follows from standard theory (see for example Karlin and Taylor, 1975, pp. 191 and 201) that the probability that t is in an interval $[T_n, S_n)$ for some n, tends to one as t tends to infinity. This completes the proof of the first part of the theorem. □

Proof of part (b) of Theorem 1. The key to the proof of the second part of the theorem is the following observation: If $k \geq N+2$, then

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mu_t\{\eta\colon \eta(j)=1 \text{ for } n \leq j < n+k\} \\
&\quad = \sum_{j=n}^{n+k-1} \mu_t\{\eta\colon \eta(j)=0,\ \eta(i)=1 \text{ for } n \leq i < n+k,\ i \neq j\} \\
&\quad\quad - \mu_t\{\eta\colon \eta(i)=0 \text{ for } n-N \leq i < n \text{ and } \eta(j)=1 \text{ for } n \leq j < n+k\} \\
&\quad\quad - \mu_t\{\eta\colon \eta(j)=1 \text{ for } n \leq j < n+k \text{ and} \\
&\qquad\qquad \eta(i)=0 \text{ for } n+k \leq i < n+k+N\}
\end{aligned} \tag{2.1}$$

for any n.

Assume now that μ is translation invariant and puts no mass on the configurations $\eta \equiv 0$ and $\eta \equiv 1$. First we will show that the limit of μ_t exists. Note that μ_t is translation invariant for each t, and hence the first and last terms in the sum are $\geq$ the two negative terms respectively. Therefore the right side of (2.1) can be written as a sum of nonnegative terms. Hence the left side of (2.1) is integrable in t. It then follows that each of the nonnegative expressions on the right of (2.1) is integrable as well, and hence we conclude that

$$\int_0^\infty \mu_t\{\eta\colon \eta(j)=0,\ \eta(i)=1 \text{ for } j-l \leq i \leq j+k,\ i \neq j\}\,\mathrm{d}t < \infty \tag{2.2}$$

for $k, l \geq 1$, $k+l \geq N+1$. Since the process is invariant under interchange of 0's and 1's, in any statement such as (2.2), the roles of 0 and 1 may be interchanged.

Next, we wish to prove that many other cylinder probabilities are integrable. This will be done via a series of reductions. In carrying out these reductions, it is important to keep in mind that if one cylinder probability is integrable, then so is any other cylinder probability which leads to it in one step with a positive rate. This argument will be spelled out the first time it occurs, but will just be mentioned briefly in future applications.

For $k, l \geqslant 1$, let

$$f_{k,l}(t)=\mu_t\{\eta\colon \eta(0)=1,\ \eta(i)=0 \text{ for } 0<i\leqslant k,\ \eta(i)=1 \text{ for } k<i\leqslant k+l, \text{ and } \eta(k+l+1)=0\}.$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}f_{k,l}(t)\geqslant f_{k-1,l+1}(t)-(4N+2)f_{k,l}(t)$$

for $k>N$. Integrating this gives

$$\int_0^\infty f_{k-1,l+1}(t)\,\mathrm{d}t\leqslant 1+(4N+2)\int_0^\infty f_{k,l}(t)\,\mathrm{d}t. \tag{2.3}$$

Interchanging the roles of k and l gives

$$\int_0^\infty f_{k+1,l-1}(t)\,\mathrm{d}t\leqslant 1+(4N+2)\int_0^\infty f_{k,l}(t)\,\mathrm{d}t \tag{2.4}$$

for $l>N$. By (2.2) and its analogue with 0's and 1's interchanged, $f_{k,1}(t)$ and $f_{1,l}(t)$ are integrable if $k, l\geqslant N$. Therefore by (2.3) and (2.4), $f_{k,l}(t)$ is integrable if $\max\{k, l\}\geqslant N$. For $k\geqslant N$, write

$$\begin{aligned}&\mu_t\{\eta\colon \eta(0)=1,\ \eta(i)=0 \text{ for } 0<i\leqslant k, \text{ and } \eta(k+1)=1\}\\ &\quad=\sum_{j=1}^{N-1} f_{k,j}(t)+\mu_t\{\eta\colon \eta(0)=1,\ \eta(i)=0 \text{ for } 0<i\leqslant k,\\ &\qquad\qquad \eta(i)=1 \text{ for } k<i\leqslant k+N\}.\end{aligned} \tag{2.5}$$

All of the terms in the sum are integrable by the earlier observations. Similarly, the last term on the right of (2.5) is integrable, via a series of reductions, by comparison with

$$\mu_t\{\eta\colon \eta(0)=1,\ \eta(1)=0 \text{ and } \eta(i)=1 \text{ for } 2\leqslant i\leqslant k+N\}.$$

This cylinder probability is in turn integrable by (2.2). Thus, we conclude that the cylinder probability on the left of (2.5) is integrable for $k\geqslant N$.

Now take $k\geqslant N$ (this restriction is in force for this entire paragraph), and consider any cylinder probability of the form

$$\mu_t\{\eta\colon \eta(i)=\varepsilon(i) \text{ for } -N<i\leqslant k+N\},$$

where $\varepsilon(0)=\varepsilon(k+1)=1$, $\varepsilon(i)=0$ for some $1\leqslant i\leqslant k$, and is otherwise arbitrary. Such a cylinder probability can be compared with either

$$\mu_t\{\eta\colon \eta(i)=\varepsilon(i) \text{ for } -N<i\leqslant 0 \text{ and } k<i\leqslant k+N, \text{ and } \eta(i)=0 \text{ for } 1\leqslant i\leqslant k\}$$

or with

$$\mu_t\{\eta\colon \eta(i)=\varepsilon(i) \text{ for } -N<i\leqslant 0 \text{ and } k<i\leqslant k+N,\ \eta(j)=0, \text{ and } \eta(i)=1 \text{ for } 1\leqslant i\leqslant k,\ i\neq j\}$$

for some $1 \leqslant j \leqslant k$. To see this, use the lemma to show that the configuration ε is flippable on $I = [1, k]$ to either $\varepsilon^{0,I}$ or to $\varepsilon^{1,I}$. In the first case, the comparison is made with the first probability above. In the second case, removing the last flip shows that ε is flippable on $I = [1, k]$ to a configuration which has exactly one zero in $[1, k]$, so the comparison is with the second probability above. But each of these two cylinder probabilities is integrable, since they are smaller than those considered in (2.5) and (2.2) respectively.

It now follows easily that any cylinder probability which is not of the form $\cdots 1\,1\,1\,0\,0 \cdots$ or $\cdots 0\,0\,1\,1\,1 \cdots$ (i.e., in which there is at least one change from 1 to 0 and at least one from zero to one) is integrable. To see this, we can assume without loss of generality (since any cylinder probability with constraints at m sites can be written as the sum of 2^p cylinder probabilities with constraints at $m+p$ sites), that it is of the form

$$\mu_t\{\eta\colon \eta(i) = \varepsilon(i) \text{ for } 0 < i \leqslant k + N\},$$

for some $k > 2$, where $\varepsilon(i)$ is arbitrary except that there are $0 < j < l < m \leqslant k$ so that

$$\varepsilon(j) = \varepsilon(m) \neq \varepsilon(l).$$

If $\varepsilon(k+N) = \varepsilon(l)$, write

$$\mu_t\{\eta\colon \eta(i) = \varepsilon(i) \text{ for } 0 < i \leqslant k + N\} \leqslant \mu_t\{\eta\colon \eta(i) = \varepsilon(i) \text{ for } l \leqslant i \leqslant k + N\},$$

while if $\varepsilon(k+N) \neq \varepsilon(l)$, write

$$\mu_t\{\eta\colon \eta(i) = \varepsilon(i) \text{ for } 0 < i \leqslant k + N\} \leqslant \mu_t\{\eta\colon \eta(i) = \varepsilon(i) \text{ for } j \leqslant i \leqslant k + N\}.$$

In either case, the right side is integrable by the result of the previous paragraph.

The final step in the proof that the limit of μ_t exists is to write the derivative of the cylinder probability

$$g(t) = \mu_t\{\eta\colon \eta(i) = \varepsilon(i) \text{ for } 0 < i \leqslant k\}$$

as a sum of other cylinder probabilities minus $kg(t)$. It follows that every cylinder probability which is not of the form $\cdots 1\,1\,1\,0\,0 \cdots$ or $\cdots 0\,0\,1\,1\,1 \cdots$ has a limit as $t \to \infty$. To check that the remaining cylinder probabilities have limits as well, it is enough to show this for

$$\mu_t\{\eta\colon \eta(i) = 1 \text{ for } 0 < i \leqslant k\}$$

for any $k \geqslant 1$. We will do this by showing that its derivative is integrable. This derivative consists of terms which are now known to be integrable, except for the terms

$$\begin{aligned}
&\mu_t\{\eta\colon \eta(i) = 0 \text{ for } -N < i \leqslant 1,\ \eta(j) = 1 \text{ for } 1 < j \leqslant k\} \\
&\quad + \mu_t\{\eta\colon \eta(i) = 1 \text{ for } 1 \leqslant i < k,\ \eta(j) = 0 \text{ for } k \leqslant j \leqslant k + N\} \\
&\quad - \mu_t\{\eta\colon \eta(i) = 0 \text{ for } -N < i < 1,\ \eta(j) = 1 \text{ for } 1 \leqslant j \leqslant k\} \\
&\quad - \mu_t\{\eta\colon \eta(i) = 1 \text{ for } 1 \leqslant i \leqslant k,\ \eta(j) = 0 \text{ for } k < j \leqslant k + N\}.
\end{aligned}$$

These differ by terms which are known to be integrable from

$$\mu_t\{\eta\colon \eta(1)=0,\ \eta(2)=1\}+\mu_t\{\eta\colon \eta(k-1)=1,\ \eta(k)=0\}$$
$$-\mu_t\{\eta\colon \eta(0)=0,\ \eta(1)=1\}-\mu_t\{\eta\colon \eta(k)=1,\ \eta(k+1)=0\}.$$

But these cancel by translation invariance. So, the limit of μ_t exists.

To show that the limit is a mixture of υ_0 and υ_1, it suffices to recall that all cylinder probabilities not of the form 1 1 1 0 0 0 or 0 0 0 1 1 1 are integrable, and hence converge to zero since their derivatives are bounded. So the limit of μ_t concentrates on configurations which are either $\equiv 1$, $\equiv 0$, or of the form 1 1 1 0 0 0 or 0 0 0 1 1 1. Since the limit is translation invariant, it can put no mass on these latter configurations. Thus the proof of part (b) of the theorem is complete. □

3. Threshold = Range = 1

This section is devoted to the proof of Theorem 2 and its corollary. Hence we consider the case $T=N=1$, and assume that we have a right continuous version of the threshold voter model η_t, constructed as indicated in the remark in Section 2 of Cox and Durrett (1991). This construction makes it possible to follow the trajectories of the edges of blocks of spins with the same value. In what follows, it is important to keep in mind that no new blocks are created as the process evolves. This is a special feature of the case $T=N=1$.

Proof of part (a) of Theorem 2. For each $\eta\in\{0,1\}^{\mathbb{Z}}$, define a process $X_\eta(\cdot)$ on $\mathbb{Z}\cup\{\infty\}$ as follows:

$$X_\eta(0)=\begin{cases}0 & \text{if } \eta(0)=1 \text{ and } \eta(1)=0,\\ \infty & \text{otherwise,}\end{cases}$$

and for $t>0$,

$$X_\eta(t)=\begin{cases}y+1 & \text{if } X_\eta(t-)=y\in Z \text{ and } \eta_t(y+1)=1 \text{ and } \eta_t(y+2)=0,\\ y-1 & \text{if } X_\eta(t-)=y\in Z \text{ and } \eta_t(y-1)=1 \text{ and } \eta_t(y)=0,\\ y & \text{if } X_\eta(t-)=y\in Z \text{ and } \eta_t(y)=1 \text{ and } \eta_t(y+1)=0,\\ \infty & \text{otherwise.}\end{cases}$$

Define also a process $Y_\eta(\cdot)$ on $\{0,1\}$ by

$$Y_\eta(t)=\begin{cases}1 & \text{if } X_\eta(t)\in Z \text{ and } \eta_t(X_\eta(t)+2)=1,\\ 0 & \text{otherwise.}\end{cases}$$

The interpretation of these processes is the following: If at time 0, site 0 is the right edge of a block of ones for η, then $X_\eta(t)$ is the position at time t of that same edge, assuming that it still exists, i.e., that the block of ones corresponding to that edge has not disappeared and has not joined the next block of ones to its right. In all other cases, $X_\eta(t)=\infty$. On the other hand, $Y_\eta(t)=1$ if and only if $X_\eta(t)\in Z$ and the block of zeros immediately to the right of $X_\eta(t)$ is of length one.

Let $\tau_\eta = \inf\{t\colon Y_\eta(t) = 1\}$. Since blocks of zeros of length one disappear at rate 1, we have

$$E\left(\int_0^\infty Y_\eta(t)\,\mathrm{d}t \,\middle|\, \tau_\eta < \infty\right) \le 1 \quad \text{for all } \eta \in \{0, 1\}^{\mathbb{Z}},$$

and hence

$$E\int_0^\infty Y_\eta(t)\,\mathrm{d}t \le P(\tau_\eta < \infty). \tag{3.1}$$

Let μ_t be the distribution at time t for the threshold voter model with initial distribution υ_ρ. In what follows, we take advantage of the translation invariance of the distribution to write cylinder events more economically. The expression $\mu_t(1\ 0\ 1)$, for example, denotes the probability of the event $\{\eta\colon \eta(k) = 1,\ \eta(k+1) = 0,\ \eta(k+2) = 1\}$ at time t, which is independent of k.

For $\eta \in \{0, 1\}^{\mathbb{Z}}$, let $\delta_{\eta,t}$ be the distribution at time t of the process with initial configuration η. Let T_k be the operator which translates elements of $\{0, 1\}^{\mathbb{Z}}$ k units to the right:

$$T_k(\eta)(j) = \eta(j-k).$$

Any $1\ 0$ at time t can be traced back to some $1\ 0$ at time zero. So, summing over the possible locations of the $1\ 0$ at time zero, and using the translation invariance of the evolution of the process, we see that

$$\delta_{\eta,t}\{\zeta\colon \zeta(-1) = 1, \zeta(0) = 0, \zeta(1) = 1\} = \sum_{k\in\mathbb{Z}} P(Y_{T_k(\eta)}(t) = 1, X_{T_k(\eta)}(t) = k-1)$$

for all initial configurations η. Integrating with respect to υ_ρ yields

$$\mu_t(1\ 0\ 1) = \sum_{k\in\mathbb{Z}} \int P(Y_{T_k(\eta)}(t) = 1, X_{T_k(\eta)}(t) = k-1)\upsilon_\rho(\mathrm{d}\eta),$$

and since υ_ρ is invariant under T_k, we get

$$\mu_t(1\ 0\ 1) = \sum_{k\in\mathbb{Z}} \int P(Y_\eta(t) = 1, X_\eta(t) = k-1)\upsilon_\rho(\mathrm{d}\eta).$$

Therefore, we have the following identity:

$$\int EY_\eta(t)\upsilon_\rho(\mathrm{d}\eta) = \mu_t(1\ 0\ 1).$$

Integrating this identity and using Tonelli's Theorem yields

$$\int\left(E\int_0^\infty Y_\eta(t)\,\mathrm{d}t\right)\upsilon_\rho(\mathrm{d}\eta) = \int_0^\infty \mu_t(1\ 0\ 1)\,\mathrm{d}t. \tag{3.2}$$

The key step in the proof of the first part of Theorem 2 is the observation that the expressions in (3.2) are $o(\rho)$ as $\rho \downarrow 0$. First we show that this is sufficient, by looking at the right side of (3.2). Then we will come back and check the claim by looking at the left side.

To show that part (a) of the theorem is a consequence of the fact that the right side of (3.2) is $o(\rho)$, start by writing

$$\frac{d}{dt}\mu_t(1)=\mu_t(1\,0\,0)+\mu_t(0\,0\,1)+\mu_t(1\,0\,1)-\mu_t(0\,1\,0)-\mu_t(0\,1\,1)-\mu_t(1\,1\,0)$$

$$=\mu_t(0\,1\,0)-\mu_t(1\,0\,1), \tag{3.3}$$

where the second equality follows from the translation invariance of the distribution. Similarly,

$$\frac{d}{dt}\mu_t(1\,0)=\mu_t(1\,0\,0)+\mu_t(1\,1\,0)-2\mu_t(1\,0)$$

$$=-\mu_t(1\,0\,1)-\mu_t(0\,1\,0).$$

By part (a) of Theorem 1,

$$\lim_{t\to\infty}\mu_t(1\,0)=0.$$

Therefore we can integrate the above identity to obtain

$$\rho(1-\rho)=\int_0^\infty [\mu_t(1\,0\,1)+\mu_t(0\,1\,0)]\,dt. \tag{3.4}$$

We conclude that $\mu_t(0\,1\,0)$ and $\mu_t(1\,0\,1)$ are in $L^1[0,\infty)$, and hence we may integrate (3.3) to obtain

$$D(\upsilon_\rho)=\rho+\int_0^\infty \mu_t(0\,1\,0)\,dt-\int_0^\infty \mu_t(1\,0\,1)\,dt,$$

which when combined with (3.4) gives

$$D(\upsilon_\rho)=2\rho-\rho^2-2\int_0^\infty \mu_t(1\,0\,1)\,dt.$$

Part (a) of Theorem 2 follows from this and the fact that the integral on the right side of (3.2) is $o(\rho)$.

Next, we will show that the left side of (3.2) is $o(\rho)$. Let

$$A=\{\eta:\ \eta(0)=1 \text{ and } \eta(1)=0\}$$

and for a positive integer n, let

$$B_n=\{\eta:\ \eta(i)=0 \text{ for all } i\in[-n,n]\backslash\{0\}, \text{ and } \eta(0)=1\}.$$

We will first show that

$$\lim_{n\to\infty}\sup_{\eta\in B_n}P(\tau_\eta<\infty)=0. \tag{3.5}$$

For this purpose, introduce three independent processes $U_{1,n}(t)$, $U_{2,n}(t)$ and $(V_1(t), V_2(t))$. The processes $U_{1,n}(t)$ and $U_{2,n}(t)$ are simple symmetric continuous time random walks on Z with exponential holding times of parameter one and initial states

$$U_{1,n}(0) = -n \quad \text{and} \quad U_{2,n}(0) = n.$$

The process $(V_1(t), V_2(t))$ is also a continuous time Markov process with state space $\mathbb{Z}^2 \cup \{\infty\}$, initial state $(0, 0)$ and transitions

$$\begin{aligned}
&(x, y) \to (x-1, y) \quad \text{at rate } 1,\\
&(x, y) \to (x, y+1) \quad \text{at rate } 1,\\
&(x, y) \to (x+1, y) \quad \text{at rate } 1 \text{ if } x < y,\\
&(x, y) \to (x, y-1) \quad \text{at rate } 1 \text{ if } x < y,
\end{aligned}$$

and

$$(x, x) \to \infty \quad \text{at rate } 1.$$

The process $(V_1(t), V_2(t))$ represents the evolution of the boundaries of the block of ones when the threshold voter model starts with configuration

$$\xi(x) = \begin{cases} 0 & \text{if } x \neq 0,\\ 1 & \text{if } x = 0.\end{cases}$$

We now introduce the following three stopping times:

$$\sigma_1 = \inf\{t: (V_1(t), V_2(t)) = \infty\},$$

$$\sigma_{2,n} = \inf\{t: V_1(t) = U_{1,n}(t) + 1\}$$

and

$$\sigma_{3,n} = \inf\{t: V_2(t) = U_{2,n}(t) - 2\}.$$

Statement (3.5) follows from the fact that for all $\eta \in B_n$,

$$P(\tau_\eta = \infty) \geq P(\sigma_1 < \sigma_{2,n} \text{ and } \sigma_1 < \sigma_{3,n})$$

and the observation that since $P(\sigma_1 < \infty) = 1$,

$$\lim_{n\to\infty} P(\sigma_1 < \sigma_{2,n} \text{ and } \sigma_1 < \sigma_{3,n}) = 1.$$

Now write

$$\int \left(E \int_0^\infty Y_\eta(t)\,\mathrm{d}t\right) \upsilon_\rho(\mathrm{d}\eta)$$

$$\leq \upsilon_\rho(A\setminus B_n) \sup_{\eta\in A\setminus B_n} E \int_0^\infty Y_\eta(t)\,\mathrm{d}t + \upsilon_\rho(B_n) \sup_{\eta\in B_n} E \int_0^\infty Y_\eta(t)\,\mathrm{d}t.$$

By (3.1), the right side of this inequality is bounded above by

$$\rho(1-\rho)2n\rho + \rho \sup_{\eta\in B_n} P(\tau_\eta < \infty).$$

This is $o(\rho)$ by (3.5), and the proof of the first part of Theorem 2 is complete. □

In proving part (b) of Theorem 2 and the corollary, we need to use some properties of

$$f_n(\rho) = P(X_n \text{ is odd}),$$

where X_n is binomially distributed with parameters n and ρ. These properties are most easily seen from $f_0(\rho) = 0$ and the recursion

$$f_{n+1}(\rho) = (1-\rho)f_n(\rho) + \rho[1 - f_n(\rho)],$$

or equivalently,

$$2f_{n+1}(\rho) - 1 = (1-2\rho)[2f_n(\rho) - 1]. \tag{3.6}$$

By induction, one then proves the following statements for $n \geqslant 0$:

$$|2f_n(\rho) - 1| \leqslant |1-2\rho|^n. \tag{3.7}$$

$$f_n'(\rho) = n[1 - 2f_{n-1}(\rho)]. \tag{3.8}$$

$$f_n''(\rho) = -2n(n-1)[1 - 2f_{n-2}(\rho)]. \tag{3.9}$$

$$f_n(\rho) \leqslant n\rho, \qquad f_n(\rho) \sim n\rho \quad \text{as } \rho \downarrow 0. \tag{3.10}$$

$$f_{n+1}(\rho) - f_n(\rho) = \rho[1 - 2f_n(\rho)] \geqslant 0 \quad \text{if } 0 \leqslant \rho \leqslant \tfrac{1}{2}. \tag{3.11}$$

Proof of the Corollary. By the duality relation (1.1), if the initial measure is υ_ρ, then the distribution at time t of the threshold voter model satisfies

$$\mu_t(1) = \sum_{n=0}^{\infty} \pi_t(2n+1) f_{2n+1}(\rho), \tag{3.12}$$

where π_t is the distribution of the DBARW at time t when it starts from a singleton. If the DBARW were not positive recurrent, it would follow that $\pi_t(1)$ tends to 0 as $t \to \infty$. (Note, however, that it does not necessarily follow that $\pi_t(k)$ tends to 0 for $k > 1$.) Passing to the limit in (3.12) and using (3.11) gives

$$D(\upsilon_\rho) \geqslant f_3(\rho)$$

for $0 \leqslant \rho \leqslant \frac{1}{2}$. By (3.10), this contradicts the first part of Theorem 2, and hence the DBARW is positive recurrent. By Scheffe's Theorem, we may pass to the limit in (3.12) to conclude that

$$D(\upsilon_\rho) = \sum_{n=0}^{\infty} \pi(2n+1) f_{2n+1}(\rho). \tag{3.13}$$

It follows from (3.10), (3.13), Fatou's Lemma, and the first part of Theorem 2 that

$$\sum_{n=0}^{\infty} (2n+1)\pi(2n+1) \leqslant 2.$$

Using this and (3.10), one can divide (3.13) by ρ and pass to the limit using the Dominated Convergence Theorem to finish the proof of the corollary. □

Proof of part (b) of Theorem 2. By the representation (3.12), it suffices to show that $f_n(\rho)$ is concave on $[0, \frac{1}{2}]$ and convex on $[\frac{1}{2}, 1]$ for odd n. By (3.9) we need to show that

$$(2\rho-1)[2f_n(\rho)-1]\geqslant 0$$

for odd n. But this is an immediate consequence of $f_1(\rho)=\rho$, (3.6) and induction. □

4. Threshold = Range ⩾ 2

The proof of Theorem 3 is based on two lemmas. Throughout the section, we assume that the threshold and range are both N.

Lemma 1. *There is a universal constant C so that if the initial distribution is υ_ρ and $0<\rho<\frac{1}{2}$, then*

$$\mu_{-N\ln\rho}(1)\leqslant CN^3(-\ln\rho)[4\rho(1-\rho)]^N$$

for all N.

Proof. Let

$$A_k=\{\eta\colon \text{for every } x\in[-k,k],\ \eta \text{ is flippable at } x \text{ if and only if } \eta(x)=1\},$$

and let Y_λ denote a random variable with a Poisson distribution with parameter λ. We wish to give an upper bound to the following expression:

$$\sup_{\eta\in A_k} P^\eta\left(\sum_{x\in[-N,N]\setminus\{0\}}\eta_s(x)\geqslant N \text{ for some } s\in[0,t]\right), \tag{4.1}$$

where $t>0$. By the definition of A_k, it is clear that an upper bound for (4.1) is given by $2P(Y_{Nt}\geqslant 1)$, since every configuration in A_k has fewer than N ones in $[-N,N]\setminus\{0\}$. To improve the bound, note that if $k>N$, then a coordinate has to flip in the region $[-2N,-N-1]\cup[N+1,2N]$ before a coordinate in state zero in $[-N,N]\setminus\{0\}$ becomes flippable. Hence in this case, an upper bound for (4.1) is given by $2P(Y_{Nt}\geqslant 2)$. Arguing inductively, we conclude that an upper bound for (4.1) is given by $2P(Y_{Nt}\geqslant k/N)$.

Let c and λ be positive reals satisfying $\ln(c/\lambda)\geqslant 2$. Then

$$P(Y_\lambda\geqslant c)\leqslant E(c/\lambda)^{Y_\lambda-c}=\mathrm{e}^{c-\lambda-c\ln(c/\lambda)}\leqslant \mathrm{e}^{-c}.$$

Combining this with the earlier upper bound for (4.1) gives the upper bound of $2\exp(-k/N)$ for the expression in (4.1) whenever $k\geqslant \mathrm{e}^2N^2t$. So, for such values of k and t, we conclude that

$$\sup_{\eta\in A_k:\,\eta(0)=0} P^\eta(\eta_t(0)=1)\leqslant 2\,\mathrm{e}^{-k/N}. \tag{4.2}$$

Suppose now that $\eta \in A_k$ and $\eta(0) = 1$. Then, as long as

$$\sum_{x \in [-N,N]\setminus\{0\}} \eta_s(x) < N,$$

the spin at 0 flips to zero at rate 1 and does not flip back to one. Therefore,

$$\sup_{\eta \in A_k:\, \eta(0)=1} P^\eta(\eta_t(0) = 1) \le 2\,\mathrm{e}^{-k/N} + \mathrm{e}^{-t}. \tag{4.3}$$

Combining (4.2) and (4.3) yields

$$\sup_{\eta \in A_k} P^\eta(\eta_t(0) = 1) \le 2\,\mathrm{e}^{-k/N} + \mathrm{e}^{-t} \tag{4.4}$$

for $k \ge \mathrm{e}^2 N^2 t$. Now write

$$\mu_t(1) \le \upsilon_\rho(A_k^c) + \sup_{\eta \in A_k} P^\eta(\eta_t(0) = 1). \tag{4.5}$$

Since υ_ρ is translation invariant and

$$\upsilon_\rho\left(\left\{\eta : \sum_{x \in [-N,N]\setminus\{0\}} \eta(x) \ge N\right\}\right) = \sum_{j=N}^{2N} \binom{2N}{j} \rho^j (1-\rho)^{2N-j} \le [4\rho(1-\rho)]^N$$

for $0 < \rho < \frac{1}{2}$, we have

$$\upsilon_\rho(A_k^c) \le (2k+1)[4\rho(1-\rho)]^N. \tag{4.6}$$

Combining (4.4), (4.5) and (4.6) yields

$$\mu_t(1) \le 2\,\mathrm{e}^{-k/N} + \mathrm{e}^{-t} + (2k+1)[4\rho(1-\rho)]^N$$

whenever $k \ge \mathrm{e}^2 N^2 t$. The statement of the lemma follows from this by taking

$$t = -N \ln \rho \quad \text{and} \quad k = [-\mathrm{e}^2 N^3 \ln \rho] + 1. \qquad \square$$

Lemma 2. *If the initial distribution μ is translation invariant, then*

$$\mu_t(1) \le (N+2)\mu(1)$$

for all $t \ge 0$.

Proof. Consider the following modification of the threshold voter model:

(a) If the spin at coordinate x is equal to one, then it flips to zero at rate one if at least one of the neighboring spins is zero and at least N spins in the interval $[x-N, x+N]$ are zero. Otherwise, its flip rate is zero.

(b) If a block of zero spins is of length k and $k \le N+1$, then all the spins in this block flip simultaneously to one at rate k. If a block of zero spins is of length k and $k > N+1$, then the spins in this block behave as they do in the threshold voter model. In particular, the interior spins cannot flip.

We will next use coupling techniques to show that it suffices to prove the statement of the lemma for this new model. The idea is to couple together the threshold voter model η_t with the new model ζ_t in such a way that the coordinatewise inequality $\eta_t \le \zeta_t$ is preserved. The joint transitions for (η_t, ζ_t) are taken to be the following:

1. $(1,1)\to(0,0)$ at site x at rate one if the conditions for flipping of $\zeta_t(x)$ are satisfied;

2. $(1,1)\to(0,1)$ at site x at rate one if the conditions for flipping of $\eta_t(x)$ are satisfied, but those for flipping of $\zeta_t(x)$ are not;

3. $(0,1)\to(1,1)$ at site x at rate one if the conditions for flipping of $\eta_t(x)$ are satisfied;

4. $(0,1)\to(0,0)$ at site x at rate one if the conditions for flipping of $\zeta_t(x)$ are satisfied;

5. for any block of zero spins in ζ_t of length $k\leq N+1$, at rate k, that block changes entirely to ones in the process ζ_t, and the spin at a site chosen uniformly at random from that block changes to a one in the process η_t, provided that the conditions for its flipping are satisfied;

6. $(0,0)\to(1,1)$ at any site x which is the endpoint of a block of zero spins in ζ_t of length $k>N+1$ at rate one if the conditions for flipping of $\eta_t(x)$ are satisfied; and

7. $(0,0)\to(0,1)$ at any site x which is the endpoint of a block of zero spins in ζ_t of length $k>N+1$ at rate one if the conditions for flipping of $\zeta_t(x)$ are satisfied, and those for flipping of $\eta_t(x)$ are not satisfied.

In proving the statement of the lemma for the new process, it is important to notice that "blocks of zeros are not created" in the new process. Using translation invariance of the distribution, this observation leads to the cancellation of a number of terms which would otherwise have appeared on the right of the following derivatives:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu_t(1\,0)\leq-\sum_{k=1}^{N+1}k\mu_t(1\overbrace{0\,0\cdots0\,0}^{k}1).$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mu_t(1)\leq{}&\sum_{k=1}^{N+1}k^2\mu_t(1\overbrace{0\,0\cdots0\,0}^{k}1)+\mu_t(\overbrace{1\,1\cdots1\,1}^{N}\overbrace{0\,0\cdots0\,0}^{N+2})\\&+\mu_t(\overbrace{0\,0\cdots0\,0}^{N+2}\overbrace{1\,1\cdots1\,1}^{N})-\mu_t(1\overbrace{0\,0\cdots0\,0}^{N})\\&-\mu_t(\overbrace{0\,0\cdots0\,0}^{N}1)\\\leq{}&\sum_{k=1}^{N+1}k^2\mu_t(1\overbrace{0\,0\cdots0\,0}^{k}1)\leq(N+1)\sum_{k=1}^{N+1}k\mu_t(1\overbrace{0\,0\cdots0\,0}^{k}1).\end{aligned}$$

Combining these inequalities, one sees that

$$\mu_t(1)+(N+1)\mu_t(1\,0)$$

is nonincreasing in t. Therefore

$$\mu_t(1)\leq\mu_t(1)+(N+1)\mu_t(1\,0)\leq\mu(1)+(N+1)\mu(1\,0)\leq(N+2)\mu(1),$$

which completes the proof of the lemma. $\square$

Proof of Theorem 3. Let μ_t be the distribution at time t when the initial distribution is υ_ρ, where $0<\rho<\frac{1}{2}$. Using Lemma 1, and applying Lemma 2 to the process with initial distribution $\mu_{-N\ln\rho}$, we see that for $t\geq -N\ln\rho$,

$$\mu_t(1)\leq C(N+2)N^3(-\ln\rho)[4\rho(1-\rho)]^N,$$

and hence

$$D_N(\upsilon_\rho)\leq C(N+2)N^3(-\ln\rho)[4\rho(1-\rho)]^N.$$

Parts (a) and (c) of Theorem 3 follow immediately from this bound.

To prove part (b), note that for any $t>0$,

$$\mu_t\{\eta\colon \eta(j)=1 \text{ for } 0\leq j\leq N+1\}\geq \rho^N(1-\rho)^2\,\mathrm{e}^{-(N+2)t}[1-\mathrm{e}^{-t}]^2. \tag{4.7}$$

The left side of (4.7) is nondecreasing in t by (2.1) and the remarks following it. The right side is maximized by t such that $\mathrm{e}^{-t}=(N+2)/(N+4)$. Therefore,

$$D_N(\upsilon_\rho)\geq \rho^N(1-\rho)^2\frac{4(N+2)^{N+2}}{(N+4)^{N+4}}.$$

The result follows immediately from this. □

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